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Optimality Conditions for D.C. Vector Optimization Problems Under Reverse Convex Constraints

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Abstract. In this paper, we establish global necessary and sufficient optimality conditions for D.C. vector optimization problems under reverse convex constraints. An application to vector fractional mathematical programming is also given.

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1. Introduction

In very recent years, the analysis and applications of D.C. mappings (difference of convex mappings) have been of considerable interest [11, 18, 27, 31]. Generally, nonconvex mappings that arise in nonsmooth optimization are often of this type. Recently, extensive work on the analysis and optimization of D.C. mappings has been carried out [7, 8, 21]. However, much work remains to be done.

Reverse convex optimization, that is, minimizing an extended real-valued convex function subject to a reverse convex constraint, constitutes a general framework for a large class of nonconvex optimization problems including D.C. optimization (minimizing or maximizing a difference of two extended real-valued convex functions), maximizing a convex function over a convex set, and minimizing a convex function over a reverse convex set, i.e., the complement of a convex subset of a convex set. This subject has received increased attention in recent years mainly for numerical purposes [13, 28, 30], duality theory in D.C. optimization [15, 16, 23] or from the point of view of necessary and sufficient conditions for optimality with the use of subdifferential calculus of convex analysis and regularising techniques [6, 10, 11, 19, 20, 29].

In this paper, we are concerned with the multiobjective optimization problem

$$(P)\begin{cases} Y^+ - \text{Minimize } f(x) - g(x) \\ \quad \text{subject to : } x \in X \setminus S \end{cases}$$

where X and Y are Banach spaces, $f, g: X \to Y$ are Y^+-convex, proper and lower semi-continuous mappings, S is a nonempty open convex subset of X, and $Y^+ \subset Y$ is a pointed, convex and closed cone with nonempty interior. Our approach consists of using a special scalarization function introduced in optimization by Hiriart-Urruty [10] to detect necessary and sufficient optimality conditions for (P). Here, convex analysis theory plays a crucial role in our investigation.

Applying Corollary 3.3 and Theorem 3.4, we deduce optimality conditions for the special multiobjective optimization problem

$$\begin{array}{l} \mathbb{R}^p_+ - \text{Minimize } \left(\dfrac{f_1(x)}{g_1(x)}, \ldots, \dfrac{f_p(x)}{g_p(x)} \right) \\ \quad \text{subject to : } h(x) \notin -int Z^+ \end{array}$$

where $f_1, \ldots, f_p, g_1, \ldots, g_p: X \to \mathbb{R}$ are lower semicontinuous functions such that

$$f_i(x) \geqslant 0 \quad \text{and} \quad g_i(x) > 0 \quad \text{for all } i = 1, \ldots, p$$

Z^+ is a nonempty closed convex cone and h is a Z^+-convex mapping defined from X into another Banach space Z.

The rest of the paper is written as follows : Section 2 contains basic definitions and preliminary material. Section 3 is devoted to main results (optimality conditions). Section 4 discusses an application to vector fractional mathematical programming with reverse convex constraints.

2. Preliminaries

Throughout this paper, $X, Y, Z$and W are Banach spaces whose topological dual spaces are X^*, Y^*, Z^* and W^*, respectively. Let $Y^+ \subset Y$ (respectively $Z^+ \subset Z$) be a pointed $\left(Y^+ \cap \left(-Y^+\right) = \{0\}\right)$, convex and closed cones $\left(\lambda Y^+ \subset Y^+ \text{ for all } \lambda \geqslant 0\right)$ with nonempty interior introducing a partial order in Y (respctively in Z) defined by

$$y_1 \leqslant_Y y_2 \Leftrightarrow y_2 \in y_1 + Y^+.$$

We adjoin to Y tow artificial elements $+\infty$ and $-\infty$ such that

$$-\infty = -(+\infty), \, y_1 - \infty \leqslant_Y y_2 \quad \text{for all } y_1, y_2 \in Y.$$

Moreover

$$y_2 \leqslant_Y y_1 + \infty = +\infty \quad \text{for all } y_1, y_2 \in Y \cup \{+\infty\}.$$

The negative polar cone Y° of Y^+ is defined as

$$Y^\circ = \{y^* \in Y^* : \langle y^*, y \rangle \leqslant 0 \quad \text{for all } y \in Y^+\}$$

where $\langle ., . \rangle$ is the dual pairs.

Since convexity plays an important role in the following investigations, recall the concept of cone-convex mappings.
The mapping $f: X \to Y \cup \{+\infty\}$ is said to be Y^+-convex if for every $\alpha \in [0, 1]$ and $x_1, x_2 \in X$

$$\alpha f(x_1) + (1-\alpha) f(x_2) \in f(\alpha x_1 + (1-\alpha) x_2) + Y^+.$$

DEFINITION 2.1. A mapping $h: X \to Y \cup \{+\infty\}$ is said to be Y^+-D.C. if there exists two Y^+-convex mappings f and g such that

$$h(x) = f(x) - g(x) \quad \forall x \in X.$$

Let us recall the definition of the lower semicontinuity of a mapping. For more details on this concept, we refer the interested reader to [4, 22].

DEFINITION 2.2. [22] A mapping $f: X \to Y \cup \{+\infty\}$ is said to be lower semicontinuous at $\bar{x} \in X$, if for any neighborhood V of zero and for any $b \in Y$ satisfying $b \leqslant_Y f(\bar{x})$, there exists a neighborhood U of $\bar{x}$ in X such that

$$f(U) \subset b + V + (Y^+ \cup \{+\infty\}).$$

DEFINITION 2.3. [24, 32] Let $f: X \to Y \cup \{+\infty\}$ be a Y^+-convex mapping. The vectorial subdifferential of f at $\overline{x} \in \text{dom} f$ is given by

$$\partial^v f(\overline{x}) = \{T \in L(X, Y) : T(h) \leqslant_Y f(\overline{x} + h) - f(\overline{x}) \forall h \in X\}.$$

REMARK 2.1. When f is a convex function, $\partial^v f(\overline{x})$ reduces to the well-known subdifferential

$$\partial f(\overline{x}) = \partial_{A.C} f(\overline{x}) = \{x^* \in X^* : f(x) - f(\bar{x}) \geqslant \langle x^*, x - \overline{x} \rangle \quad \text{for all } x \in X\}.$$

REMARK 2.2. Let $f: X \to Y \cup \{+\infty\}$ be a Y^+–convex mapping. If f is also continuous at $\overline{x}$, then

$$\partial^v f(\overline{x}) \neq \emptyset.$$

The next concept was introduced in [5] in finite dimension. We give it in the infinite case.

DEFINITION 2.4. Let U be a nonempty subset of Y. A functional $g: U \to \mathbb{R} \cup \{+\infty\}$ is called Y^+-increasing on U, if for each $y_0 \in U$

$$y \in \left(y_0 + Y^+\right) \cap U \quad \text{implies } g(y) \geqslant g(y_0).$$

In [14], and using the separation Hahn-Banach geometric theorem, B. Lemaire set the following proposition which generalize both Gol'shtein's result [9] and Levin's result [17]. He used, for a simple application $h: Y \to \mathbb{R} \cup \{+\infty\}$, and another application which is Y^+-increasing $g: Y \to \mathbb{R} \cup \{+\infty\}$, the convention that

$$g \circ h(x) = g(h(x)) \text{ if } h(x) \in \operatorname{dom}(g) \quad \text{and} \quad g(+\infty) = +\infty.$$

Consequently, $g \circ h$ is an application from X into $\mathbb{R} \cup \{+\infty\}$ and its effective domain is given by

$$\operatorname{dom}(g \circ h) = \operatorname{dom}(h) \cap h^{-1} \operatorname{dom}(g).$$

PROPOSITION 2.1. [14] *Let X and Y be two real Banach spaces. Consider a mapping h from X into* $Y \cup \{+\infty\}$ *and a function g from Y into* $\mathbb{R} \cup \{+\infty\}$. *If*

(i) *h is* Y^+-*convex,*
(ii) *g is convex,* Y^+-*increasing and continuous in some point of* $h(X)$.

Then

$$\partial(g \circ h)(x) = \bigcup_{y^* \in \partial g(h(x))} \partial\left(y^* \circ h\right)(x).$$

In the sequel, we shall need the following result of [4]. Under the nonemptiness of the set $\left\{x \in X: h(x) \in -\text{int } Y^+\right\}$, one has

$$\partial(\delta_{-Y^+} \circ h)(\bar{x}) = \bigcup_{\substack{y^* \in (-Y^+)^\circ \\ \langle y^*, h(\bar{x})\rangle = 0}} \partial\left(y^* \circ h\right)(\bar{x}) \tag{2.1}$$

where the symbol $\langle , \rangle$ denotes the bilinear pairing between Y and Y^*, and δ_S is the indicator function of S.

REMARK 2.3. Notice that the function $y \to \delta_{-Y^+}(y)$ is Y^+-increasing. Moreover for any Y_+-convex mapping $h: X \to Y \cup \{+\infty\}$, the composite function $\delta_{-Y^+} \circ h$ is also convex.

For a subset A of Y, we consider the function

$$\Delta_A(y) = \begin{cases} d(y, A) & \text{if } y \in Y \backslash A \\ -d(y, Y \backslash A) & \text{if } y \in A \end{cases}$$

where $d(y, A) = \inf\{\|u - y\| : u \in A\}$. This function was introduced by Hiri-art-Urruty [10] (see also [12]), and used after by Ciligot-Travain [2], and Amahroq and Taa [1].
The next proposition has been established by Hiriart-Urruty [10].

PROPOSITION 2.2. [10] *Let $A \subset Y$ be a pointed closed convex cone with nonempty interior and $A \neq Y$. The function Δ_A is convex, positively homogeneous, 1-Lipschitzian, decreasing on Y with respect to the order introduced by S. Moreover* $(Y \backslash A) = \{y \in Y : \Delta_A(y) > 0\}$, $\operatorname{int}(A) = \{y \in Y : \Delta_A(y) < 0\}$ *and the boundary of* $A: bd(A) = \{y \in Y : \Delta_A(y) = 0\}$.

It is easy to verify the following lemma.

LEMMA 2.3. *The function Φ: $Y \to \mathbb{R}$ defined by*

$$\Phi(y) = \Delta_{-\operatorname{int}(Y^+)}(y)$$

is (Y^+)-increasing on Y.

Let K be a closed convex subset of X. The normal cone $N(K, \bar{x})$ to K at $\bar{x}$ is denoted by

$$N(K, \bar{x}) = \left\{x^* \in X^* : 0 \geqslant \langle x^*, x - \bar{x} \rangle \text{ for all } x \in K\right\}.$$

This cone can be also written as

$$N(K, \bar{x}) = \partial \delta_K(\bar{x})$$

where δ_K is the indicator function of K. Properties of the subdifferential and the normal cone can be found in Rockafellar [25].
As a direct consequence of Proposition 2.2, one has the following result.

PROPOSITION 2.4. [2] *Let $A \subset Y$ be a closed convex cone with a nonempty interior. For all $y \in Y$. one has*

$$0 \notin \partial \Delta_S(y).$$

3. Optimality Conditions

We begin by giving necessary optimality condition for the optimization problem

$$(P_1): \begin{cases} Y^+ - \text{Minimize } f(x) - g(x) \\ \text{subject to:} \qquad x \in C, \end{cases}$$

where $f, g: X \to Y \cup \{+\infty\}$ are convex and lower semi-continuous mappings and C a closed set.

The point $\overline{x} \in C$ is an efficient (respectively weak efficient) solution of (P_1) if $(f-g)(\overline{x})$ is a Pareto (respectively weak Pareto) minimal vector of $(f-g)(C)$.

For all the sequel, we assume that $\overline{x} \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$.

LEMMA 3.1. *If $\bar{x} \in C$ is a weak minimal solution of (P_1) with respect to Y^+, then for all $T \in \partial^v g(\bar{x})$, $\bar{x}$ solves the following scalar convex minimization problem*

$$(P_2) \begin{cases} \text{Minimize } \Delta_{-\mathrm{int}(Y^+)}(f(x) - f(\bar{x}) - T(x - \bar{x})) \\ \text{Subject to } x \in C. \end{cases}$$

Proof. Suppose the contrary. There exist $x_0 \in C$ such that

$$\Delta_{-\mathrm{int}(Y^+)}(f(x_0) - f(\bar{x}) - T(x_0 - \bar{x})) < \Delta_{-\mathrm{int}(Y^+)}(0) = 0.$$

This implies with Proposition 2.4 that

$$f(x_0) - f(\bar{x}) - T(x_0 - \bar{x}) \in -\mathrm{int}\left(Y^+\right). \tag{3.1}$$

By assumption, since $T \in \partial^v g(\bar{x})$, one has

$$\langle T, x_0 - \bar{x} \rangle \in -(g(x_0) - g(\bar{x})) - Y^+ \tag{3.2}$$

From $(3.1), (3.2)$ and the fact that $\mathrm{int}\left(Y^+\right) + Y^+ \subset \mathrm{int}\left(Y^+\right)$, we obtain

$$f(x_0) - g(x_0) - (f(\overline{x}) - g(\overline{x})) \in -\mathrm{int}\left(Y^+\right)$$

which contradicts the fact that $\bar{x}$ is a weak minimal solution of (P_1). □

THEOREM 3.2. *Assume that f is finite and continuous at $\bar{x}$. If $\bar{x}$ is a weak minimal solution of (P_1) then for all $T \in \partial^v g(\bar{x})$ there exist $y^* \in \left(-Y^+\right)^\circ \setminus \{0\}$ such that*

$$y^* \circ T \in \partial \left(y^* \circ f\right)(\overline{x}) + N^c(C, \bar{x}).$$

Here, $N^c(C, x)$ designes the Clarke normal cone to C at x.

Proof. Set $H(.) = f(.) - f(\bar{x}) - T(. - \bar{x})$.

- On the one hand, as $\Delta_{-\text{int}(Y^+)}$ is Y^+-increasing and H is Y^+-convex, then $\Delta_{-\text{int}(Y^+)} \circ H$ is convex.
- On the second hand, as $\Delta_{-\text{int}(Y^+)}$ and H is continuous, then $\Delta_{-\text{int}(Y^+)} \circ H$ is continuous.

Consequently, $\Delta_{-\text{int}(Y^+)} \circ H$ is locally Lipschitzian. Denoting by $k_0 > 0$ the Lipschitz constant of $\Delta_{-\text{int}(Y^+)} \circ H$, due to the Clarke penalization [3], there exists $k \geqslant k_0$ such that

$$0 \in \partial^c \left(\Delta_{-\text{int}(Y^+)}(H(.)) + k d_C\right)(\overline{x}).$$

Applying the sum rule [3], we obtain

$$0 \in \partial^c \left(\Delta_{-\text{int}(Y^+)}(H(.))\right)(\overline{x}) + k\partial^c d(., C).$$

Since H is Y^+-convex and $\Delta_{-\text{int}(Y^+ \times Z^+)}(.)$ is convex continuous in 0 and Y^+-increasing, due to Proposition 2.1, there exist $y^* \in \partial \Delta_{-\text{int}(Y^+)}(0)$ such that

$$0 \in \partial \left(y^* \circ H\right)(\overline{x}) + N^c(C, \bar{x}).$$

Since $\Delta_{-\text{int}(Y^+)}(.)$ is a convex function and $\Delta_{-\text{int}(Y^+)}(0) = 0$ we have for all $y \in Y$

$$\Delta_{-\text{int}(Y^+)}(y) \geqslant \left\langle y^*, y\right\rangle$$

and hence for all $y \in -Y^+$

$$\left\langle y^*, y\right\rangle \leqslant \Delta_{-\text{int}(Y^+)}(y) = -d\left(y, Y \setminus -\text{Int}\left(Y^+\right)\right) \leqslant 0.$$

That is $y^* \in \left(-Y^+\right)^\circ$. We conclude from Proposition 2.4 that $y^* \neq 0$.

Consequently, there exist $y^* \in \left(-Y^+\right)^\circ \setminus \{0\}$ satisfying

$$0 \in \partial \left(y^* \circ f + \left\langle -y^* \circ T, x - \bar{x}\right\rangle\right)(\overline{x}) + N^c(C, \bar{x}).$$

Finally, for all $T \in \partial^v g(\bar{x})$, there exist $y^* \in (-Y^+)^\circ \setminus \{0\}$ such that

$$y^* \circ T \in \partial \left(y^* \circ f\right)(\overline{x}) + N^c(C, \bar{x}). \qquad \square$$

Let S be a nonempty open convex subset of X. Setting $C:=X \backslash S$, one has Theorem 3.3 which gives necessary optimality conditions for the reverse optimization problem (P).

THEOREM 3.3. *Assume that f is finite and continuous at $\bar{x}$ and that $\bar{x}$ is a weak minimal solution of (P). Then, for all $T \in \partial^v g(\bar{x})$ there exist $y^* \in (-Y^+)^\circ \setminus \{0\}$ such that*

$$y^* \circ T \in \partial \left(y^* \circ f\right)(\overline{x}) - N(S, \bar{x}).$$

Proof. Let $T \in \partial^v g(\bar{x})$. Applying Theorem 3.2, there exist $y^* \in (-Y^+)^\circ \setminus \{0\}$ such that

$$y^* \circ T \in \partial \left(y^* \circ f\right)(\overline{x}) + N^c(C, \bar{x}). \tag{3.3}$$

Since S is an open convex subset, it is also epi-Lipschitzian at $\bar{x}$ [26]. By a result of Rockafellar [26], we conclude that

$$N^c(X \setminus S, \bar{x}) = -N(S, \bar{x}). \tag{3.4}$$

Combining (3.3) and (3.4), we get the result. □

REMARK 3.1. In Theorem 3.3, if $\overline{x}$ is an interior point of S, then for all $T \in \partial^v g(\bar{x})$ there exist $y^* \in (-Y^+)^\circ \setminus \{0\}$ such that

$$y^* \circ T \in \partial \left(y^* \circ f\right)(\overline{x}).$$

THEOREM 3.4. *Suppose that $f, g: X \to Y \cup \{+\infty\}$ are convex, proper and lower semicontinuous, S is a nonempty open convex subset of X* and $\bar{x} \in \mathrm{dom} f \cap \mathrm{dom} g$ *is a boundary point of S. If there exists $y^* \in (-Y^+)^\circ \setminus \{0\}$ such that*

$$\partial_\varepsilon \left(y^* \circ g\right)(\overline{x}) + N(S, \bar{x}) \subset \partial_\varepsilon \left(y^* \circ f\right)(\overline{x}) \quad \text{for all } \varepsilon > 0. \tag{3.5}$$

Then $\bar{x}$ is a weak minimal solution of (P_1).

Proof. As in Theorem 3.3,

$$N^c(X \setminus S, \bar{x}) = -N(S, \bar{x}).$$

Since $\partial^c d(., X \setminus S)(\bar{x}) \subset N^c(X \setminus S, \bar{x})$, inclusion (3.5) becomes

$$\partial_\varepsilon \left(y^* \circ g\right)(\overline{x}) - \partial^c d(., X \setminus S)(\bar{x}) \subset \partial_\varepsilon \left(y^* \circ f\right)(\overline{x}), \quad \text{for all } \varepsilon > 0.$$

Consequently, for all $\varepsilon > 0$

$$\partial_\varepsilon \left(y^* \circ g\right)(\overline{x}) + \partial d(., S)(\bar{x}) - \partial^c d(., X \setminus S)(\bar{x}) \subset \partial_\varepsilon \left(y^* \circ f\right)(\overline{x}) + \partial d(., S)(\bar{x}).$$

As $\partial \Delta_S(\bar{x}) \subset \partial d(., S)(\bar{x}) - \partial^c d(., X \setminus S)(\bar{x})$, we get

$$\partial_\varepsilon \left(y^* \circ g\right)(\overline{x}) + \partial \Delta_S(\bar{x}) \subset \partial_\varepsilon \left(y^* \circ f\right)(\overline{x}) + \partial d(., S)(\bar{x}) \quad \text{for all } \varepsilon > 0$$

which yields that

$$\partial_\varepsilon \left(y^* \circ g\right)(\overline{x}) + \partial \Delta_S(\bar{x}) \subset \partial_\varepsilon \left(y^* \circ f + d(., S)\right)(\overline{x}) \quad \text{for all } \varepsilon > 0. \tag{3.6}$$

Since Δ_S is convex continuous, one has

$$\partial_\varepsilon \left(y^* \circ g + \Delta_S\right)(\overline{x}) = \partial_\varepsilon \left(y^* \circ g\right)(\overline{x}) + \partial \Delta_S(\bar{x}) \quad \text{for all } \varepsilon > 0. \tag{3.7}$$

From (3.6) and (3.7), we obtain

$$\partial_\varepsilon \left(y^* \circ g + \Delta_S\right)(\overline{x}) \subset \partial_\varepsilon \left(y^* \circ f + d(., S)\right)(\overline{x}) \quad \text{for all } \varepsilon > 0.$$

By the classical Hiriart-Urruty [11] sufficient conditions, $\bar{x}$ minimize the function

$$y^* \circ f(x) - y^* \circ g(x) + d(x, X \setminus S).$$

We conclude that $\bar{x}$ (a boundary point of S) is a minimum of the problem

Minimize $y^* \circ (f(x) - g(x))$
subject to : $x \in X \setminus S$.

Finally, due to $y^* \in \left(-Y^+\right)^\circ \setminus \{0\}$, $\bar{x}$ is a weak minimal solution of (P_1). □

4. Application

In this section, we give an application to vector fractional mathematical programming under reverse convex constraints. Let $f_1, \ldots, f_p, g_1, \ldots, g_p$: $X \to \mathbb{R}$ be convex and lower semicontinuous functions such that

$$f_i(x) \geqslant 0 \text{ and } g_i(x) > 0 \quad \text{for all } i = 1, \ldots, p.$$

We denote by ϕ the mapping defined as follows:

$$\phi(x) := \left(\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_p(x)}{g_p(x)} \right).$$

We investigate the vector optimization problem

$$(P^*): \begin{cases} \mathbb{R}^p_+ - \text{Minimize } \phi(x) \\ \text{subject to : } h(x) \notin -\text{int}Z^+ \end{cases}$$

where Z^+ is a nonempty closed convex cone and h is a Z^+-convex mapping defined from X into Z.

Setting $S{:} = \{x \in X{:} h(x) \in -\text{int}Z^+\}$, we assume that $S \neq \emptyset$ and $X \backslash S \neq \emptyset$. Then we have the following results.

LEMMA 4.1. *Let $\bar{x}$ be a feasible point of problem (P^*).$\bar{x}$ is a weak minimal solution of (P^*) if and only if $\bar{x}$ is a weak minimal solution of the following problem*

$$\left(P''\right): \begin{cases} \mathbb{R}^p_+ - \text{Minimize } \left(f_1(x) - \phi_1(\bar{x}) g_1(x), \ldots, f_p(x) - \phi_p(\bar{x}) g_p(x)\right) \\ \text{subject to : } x \in X \backslash S \end{cases}$$

where $\phi_i(\bar{x}) = (f_i(\bar{x}))/(g_i(\bar{x}))$.

Proof. Let $\bar{x}$ be a weak minimal solution of (P^*). If there exists $x_1 \in \bar{x} + \mathbb{B}_X$ such that $x_1 \in X \backslash S$ and

$$(f_i(x_1) - \phi_i(\bar{x}) g_i(x_1)) - (f_i(\bar{x}) - \phi_i(\bar{x}) g_i(\bar{x})) \in -\text{Int}\left(\mathbb{R}^p_+\right).$$

Since $f_i(\bar{x}) - \phi_i(\bar{x}) g_i(\bar{x}) = 0$, one has

$$\frac{f_i(x_1)}{g_i(x_1)} - \frac{f_i(\bar{x})}{g_i(\bar{x})} \in -\text{Int}\left(\mathbb{R}^p_+\right)$$

which contradicts the fact that $\bar{x}$ is a weak minimal solution of (P^*). So $\bar{x}$ is a weak minimal solution of $\left(P''\right)$. The converse implication can be proved in a similar way. The proof is thus completed. □

LEMMA 4.2. *Denoting by $\bar{S}$ the closure in X of the subset S, we have*

$$\bar{S} := \left\{x \in X : h(x) \in -Z^+\right\}.$$

Proof. From the continuity assumption of h and the fact that the cone Y^+ is closed

$$\bar{S} \subset \left\{x \in X : h(x) \in -Z^+\right\}.$$

Conversely, let $x \in X$ such that $h(x) \in -Z^+$. From the nonemptiness of S, there exists $a \in X$ such that

$$h(a) \in -\mathrm{int}\left(Z^+\right).$$

Setting $x_n := (1/n)a + (1-(1/n))x$ for any $n \geqslant 1$, the sequence $(x_n)_{n \geqslant 1}$ converges to x.

Since h is convex, one has

$$h(x_n) \in \frac{1}{n} h(a) + \left(1 - \frac{1}{n}\right) h(x) - Z^+ \in -\mathrm{int}\left(Z^+\right) - Z^+ \subset -\mathrm{int}\left(Z^+\right);$$

which means that $x_n \in S$. Then,

$$\left\{x \in X : h(x) \in -Z^+\right\} \subset \bar{S}.$$

Finally, the desired equality holds. □

THEOREM 4.3. *Let $\overline{x}$ be a boundary point of S and assume that f is finite and continuous at $\bar{x}$. If $\bar{x}$ is a weak minimal solution of (P^*), then for all $\left(T_1, \ldots, T_p\right) \in \partial g_1(\bar{x}) \times \cdots \times \partial g_p(\bar{x})$ there exist $\left(\alpha_1^*, \ldots, \alpha_p^*\right) \in \mathbb{R}_+^p \setminus \{0\}$ and $z^* \in \left(-Z^+\right)^\circ$ such that $\langle z^*, h(\overline{x})\rangle = 0$ and*

$$\sum_{i=1}^{p} \phi_i(\overline{x}) \alpha_i^* T_i \in \partial\left(\sum_{i=1}^{p} \alpha_i^* f_i\right)(\overline{x}) - \partial\left(z^* \circ h\right)(\overline{x}).$$

Proof. Let $\left(T_1, \ldots, T_p\right) \in \partial g_1(\bar{x}) \times \cdots \times \partial g_p(\bar{x})$. Applying Lemma 4.1 and Theorem 3.3, there exist $\left(\alpha_1^*, \ldots, \alpha_p^*\right) \in \mathbb{R}_+^p \setminus \{0\}$ and $z^* \in \left(-Z^+\right)^\circ$ such that $\langle z^*, h(\overline{x})\rangle = 0$ and

$$\sum_{i=1}^{p} \phi_i(\overline{x}) \alpha_i^* T_i \in \partial\left(\sum_{i=1}^{p} \alpha_i^* f_i\right)(\overline{x}) - N(S, \bar{x}). \tag{4.1}$$

Using Lemma 4.2,

$$\delta_{\bar{S}} = \delta_{-Z^+} \circ h.$$

Since $N(S, \bar{x}) = N(\bar{S}, \bar{x})$, one obtains

$$N(S, \bar{x}) = \partial \delta_{\bar{S}}(\bar{x}) = \partial(\delta_{-Z^+} \circ h)(\bar{x}). \tag{4.2}$$

Combining (2.1), (4.1) and (4.2), we get the result. □

THEOREM 4.4. *Suppose that* $f, g: X \to Y \cup \{+\infty\}$ *are convex, proper and lower semicontinuous, S is a nonempty open convex subset of X and* $\bar{x} \in \operatorname{dom} f \cap \operatorname{dom} g$ *is a boundary point of S. Suppose also that there exists* $(\alpha_1^*, \ldots, \alpha_p^*) \in \mathbb{R}_+^p \setminus \{0\}$ *such that for any* $z^* \in (-Z^+)^\circ$ *one has* $\langle z^*, h(\overline{x}) \rangle = 0$ *and*

$$\partial_\varepsilon \left(\sum_{i=1}^{p} \phi_i(\overline{x}) \alpha_i^* g_i \right)(\overline{x}) + \partial \left(z^* \circ h \right)(\bar{x}) \subset \partial_\varepsilon \left(\sum_{i=1}^{p} \alpha_i^* f_i \right)(\overline{x}) \quad \text{for all } \varepsilon > 0. \tag{4.3}$$

Then, $\overline{x}$ is a weak minimal solution of (P^*).

Proof. As previously,

$$N_S(\overline{x}) = \partial \delta_{\overline{S}}(\overline{x}) = \partial(\delta_{-Z^+} \circ h)(\overline{x}) = \bigcup_{\substack{z^* \in (-Z^+)^\circ \\ \langle z^*, h(\bar{x}) \rangle = 0}} \partial \left(z^* \circ h \right)(\bar{x}).$$

Consequently, from inclusion (4.3), one has

$$\partial_\varepsilon \left(\sum_{i=1}^{p} \phi_i(\overline{x}) \alpha_i^* g_i \right)(\overline{x}) + N_S(\overline{x}) \subset \partial_\varepsilon \left(\sum_{i=1}^{p} \alpha_i^* f_i \right)(\overline{x}) \quad \text{for all } \varepsilon > 0.$$

Finally, applying Theorem 3.4, we finish the proof. □

EXAMPLE 4.1. Let f and $g: \mathbb{R} \to \mathbb{R}$ be given functionals with

$$f(x) = |x| \text{ and } g(x) = \frac{1}{2} x^2.$$

We consider $h: \mathbb{R} \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x \leqslant 0. \end{cases}$$

In this case, $\partial_\varepsilon g(0) = \{0\}$, $\partial_\varepsilon f(0) = [-1-\varepsilon, 1+\varepsilon]$ and $\partial h(0) = [0, 1]$. Under these assumptions, we remark that (4.3) is satisfied.

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